SHORT NOTE

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A NOTE ON A GENERALIZED M-SERIES AS A SPECIAL FUNCTION OF FRACTIONAL CALCULUS

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Abstract

In this note we like to bring audience' attention to a further extension of both Mittag-Leffler function and generalized hypergeometric function ${}_pF_q$, called generalized M-series. This is a continuation of our previous note [10] and an interesting example of the Special Functions of Fractional Calculus (SF of FC), in the sense of [7] and [2],[3], a notion that gained recently an important role in the theory of differentiation of arbitrary order and in the solutions of fractional order differential equations. We give representations of the generalized M-series in terms of the Wright generalized hypergeometric function ${}_p\Psi_q$ and Fox's H-function, and formulas for fractional calculus operators of it.

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1. The generalized M-series

To distinguish between the earlier considered M-series [10] and its new modification considered here, we call as generalized M-series the function defined by means of the power series:

$$\underset{p}{\overset{\alpha,\beta}{M}}{}_q\,(a_1,\ldots,a_p;b_1,\ldots,b_q;z) := \underset{p}{\overset{\alpha,\beta}{M}}{}_q\,(z),$$

$$= \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k}\, \frac{z^k}{\Gamma(\alpha k+\beta)}\,, \quad z,\alpha,\beta \in \mathbb{C},\ \Re(\alpha) > 0. \qquad (1)$$

Here $(a_j)_k, (b_j)_k$ are the known Pochammer symbols. As usually, the series (1) is defined when none of the parameters b_js, $j = 1, 2, \ldots, q$, is a negative integer or zero; if any numerator parameter a_j is a negative integer or zero, then the series terminates to a polynomial in z. The series in (1) is convergent for all z if $p \leq q$, it is convergent for $|z| < \delta = \alpha^\alpha$ if $p = q + 1$ and divergent, if $p > q + 1$. When $p = q + 1$ and $|z| = \delta$, the series can converge on conditions depending on the parameters. The details follow from the general theory for the Wright ${}_p\Psi_q$-functions, see e.g. in [2, p.56, Th.1.5].

2. Relations with other special functions of fractional calculus

The generalized M-series can be represented as a special case of the *Wright generalized hypergeometric function* ${}_p\Psi_q(z)$ (for its definition see e.g. [2, p.56, (1.11.14)], also [3],[4]) and of the *Fox H-function* [1] (definitions in [5], [2], [3], [4]). Namely,

$$ {}_p\overset{\alpha,\beta}{M}_q\,(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \kappa\, {}_{p+1}\Psi_{q+1}\left[\begin{array}{c} (a_1, 1), \ldots, (a_p, 1), (1, 1) \\ (b_1, 1), \ldots, (b_q, 1), (\beta, \alpha) \end{array} \middle| z \right] \tag{2} $$

$$ = \kappa\, H^{1,p+1}_{p+1,q+2}\left[-z \middle| \begin{array}{c} (1 - \alpha_j, 1; 1)_1^p, (0, 1) \\ (0, 1), (1 - \beta_j, 1)_1^q, (1 - \beta, \alpha) \end{array} \right], \quad \kappa = \prod_{j=1}^{q} \Gamma(b_j) / \prod_{j=1}^{p} \Gamma(a_j)\,. \tag{3} $$

Some special cases of the ${}_p\overset{\alpha}{M}_q\,(z)$-function are the following:

(i) For $\beta = 1$, the generalized M-series is *the M-series* from Sharma [10]:

$$ {}_p\overset{\alpha}{M}_q\,((a_j)_1^p; (b_j)_1^q; z) = {}_p\overset{\alpha,1}{M}_q\,(z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} \frac{z^k}{\Gamma(\alpha k + 1)}\,. \tag{4} $$

(ii) *The Mittag-Leffler function* (see [2]-[4], [7]): When there is no upper or lower parameters ($p = q = 0$), we have

$$ E_{\alpha,\beta}(z) = {}_0\overset{\alpha,\beta}{M}_0\,(-; -; z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad E_\alpha(z) = {}_0\overset{\alpha}{M}_0\,(-; -; z), \tag{5} $$

where $E_\alpha(z)$, $\beta = 1$ is the originally introduced *Mittag-Leffler function*, [6].

(iii) For $p = 0, q = 1, b_1 = 1$, *the Wright function* denoted by $\varphi(\alpha, \beta, z)$ ([2], p.54, (1.11.1)], [3]-[4]), or by $W(z; \alpha, \beta)$ ([7, p. 37, (1.156)] comes:

$$ \varphi(\alpha, \beta; z) = W(z; \alpha, \beta) = {}_0\Psi_1\left[\begin{array}{c} - \\ (\beta, \alpha) \end{array} \middle| z \right] \tag{6} $$

$$ = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k + \beta)} \frac{z^k}{k!} = \sum_{k=0}^{\infty} \frac{1}{(1)_k} \frac{z^k}{\Gamma(\alpha k + \beta)} = {}_0\overset{\alpha,\beta}{M}_1\,(-; 1; z). $$

(iv) *The generalized Mittag-Leffler function*, introduced by Prabhakar and studied by Kilbas et al., see [2, p. 45, (1.9.1), is obtained from (1) for $p=q=1, a=\rho\in\mathbb{C}, b=1$:

$$E^{\rho}_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{(\rho)_k}{\Gamma(\alpha k+\beta)}\frac{z^k}{k!}=\sum_{k=0}^{\infty}\frac{(\rho)_k}{(1)_k}\frac{z^k}{\Gamma(\alpha k+\beta)}={}_1\overset{\alpha,\beta}{M}_1\,(\rho;1;z). \quad (7)$$

(v) *The generalized hypergeometric function* ${}_pF_q$ (see e.g. [2]-[5]): When $\alpha=\beta=1$, with arbitrary p,q, we have (using $k!=\Gamma(1.k+1)$):

$${}_pF_q((a_j)_1^p;(b_j)_1^q;z)=\sum_{k=0}^{\infty}\frac{(a_1)_k\dots(a_p)_k}{(b_1)_k\dots(b_q)_k}\frac{z^k}{k!}={}_p\overset{1,1}{M}_q\,((a_j)_1^p;(b_j)_1^q;z). \quad (8)$$

3. Fractional integrals and derivatives of the generalized M-series

Consider the fractional *Riemann-Liouville (R-L) integral operator*, see [8] (for lower limit $a=0$), of the generalized M-series (1):

$$I_z^{\nu}\,{}_p\overset{\alpha,\beta}{M}_q\,(z)=\frac{1}{\Gamma(\nu)}\int_0^z(z-t)^{\nu-1}\left[\sum_{k=0}^{\infty}\frac{(a_1)_k\dots(a_p)_k}{(b_1)_k\dots(b_q)_k}\frac{t^k}{\Gamma(\alpha k+\beta)}\right]dt. \quad (9)$$

In a way exactly same as in the note of Sharma [10], we can calculate that

$$I_z^{\nu}\,{}_p\overset{\alpha,\beta}{M}_q\,(z)=\frac{z^{\nu}}{\Gamma(\nu+1)}\,{}_{p+1}\overset{\alpha,\beta}{M}_{q+1}\,(a_1,\dots,a_p,1;b_1,\dots,b_q,\nu+1;z). \quad (10)$$

Analogously, the *R-L fractional differential operator* (see [8]) of the generalized M-series can be considered (with $a=0$ and with respect to z):

$$D_z^{\nu}\,{}_p\overset{\alpha,\beta}{M}_q(z)=\frac{1}{\Gamma(n-\nu)}\left(\frac{d}{dz}\right)^n\int_0^z(z-t)^{n-\nu-1}\left[\sum_{k=0}^{\infty}\frac{(a_1)_k\dots(a_p)_k}{(b_1)_k\dots(b_q)_k}\frac{t^k}{\Gamma(\alpha k+\beta)}\right]dt, \quad (11)$$

where $n=[\nu]+1$. As in [10], term by term integration leads to

$$D_z^{\nu}\,{}_p\overset{\alpha,\beta}{M}_q(z)=\frac{z^{-\nu}}{\Gamma(1-\nu)}\,{}_{p+1}\overset{\alpha,\beta}{M}_{q+1}\,(a_1,\dots,a_p,1;b_1,\dots,b_q,1-\nu;z). \quad (12)$$

That is, as naturally expected for fractional calculus operators of special functions being generalized hypergeometric functions, *a R-L fractional integral or derivative of a generalized M-series is again a generalized M-series with indices p,q increased to $(p+1),(q+1)$.*

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Editorial Note. The authors submitted a manuscript containing the proofs of above relationships, following from the definitions of the SF involved or going in same way as in [10]. So we accepted to publish a short note only, aiming *to demonstrate one more interesting example of SF of FC.*

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